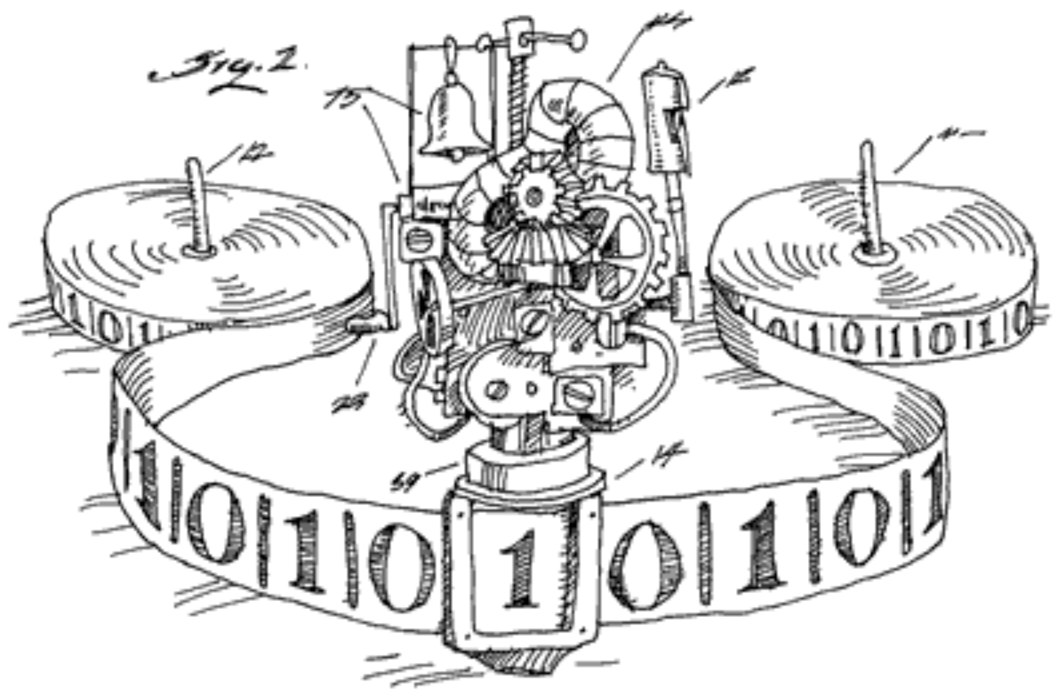
CSC4010: Introduction to Theoretical Computer Science



Tutorial 2

Question 1

1. In each of the following questions, choose either true or false.

   (a) If $A \subseteq B$, then $|A| \leq |B|$.

   ✓ True

   ◯ False

   (b) If $A \subsetneq B$, then $|A| < |B|$.

   ◯ True

   ✓ False

   (c) Every subset of a countable set is countable.

   ✓ True

   ◯ False

   (d) Every subset of a countably infinite set is countably infinite.

   ◯ True

   ✓ False

   Any $\{1,2\} \subseteq \mathbb{N}$

   (e) Every language $L \subseteq \{0,1\}^*$ is countable.

   ✓ True

   ◯ False

   (f) The set of all languages $\mathcal{L} = \{L : L \subseteq \{0,1\}^*\}$ is countable.

   ◯ True

   ✓ False

Question 2

$012 \in \mathbb{N}^3$

$n_1 n_2 n_3 \ldots n_k \in \mathbb{N}^k$

$n_i \in \mathbb{N}.$

2. Let $\mathcal{S} = \bigcup_{k \in \mathbb{N}} \mathbb{N}^k$ be the set of all finite sequences of non-negative integers, where $\mathbb{N} = \{0, 1, 2, \ldots\}$. The case $k = 0$ corresponds to the empty sequence. Show that $\mathcal{S}$ is countable using one of the following methods. You may use the fact that $\{0,1\}^*$ is countable.

    1. Construct a one-one function
    $$f : \mathcal{S} \longrightarrow \{0,1\}^*.$$
    Make sure to explain why your function is one-one.

    2. Give an enumeration of $\mathcal{S}$ in which every element has a finite position.
    Hint. For example, you may organize the sequences according to the value
    $$k + n_1 + \cdots + n_k$$
    for a sequence $(n_1, \ldots, n_k) \in \mathbb{N}^k$.

$k \in \mathbb{N}$ $\quad \langle k \rangle.$ $\quad k_1 k_2$

$\langle k_1 \rangle \langle k_2 \rangle.$

$3 \quad 4$

$\boxed{11} \boxed{1011}$

$\boxed{110} \boxed{1}$

$6 \quad 1.$

unary encoding

A number in unary:

$2 \Rightarrow 11.$

$3 \Rightarrow 111$

$4 \Rightarrow 1111$

$n \Rightarrow 1^n$

$34 \to 1110\,1111$

and in <u>general</u>: $f: S \to \{0,1\}^*$.

$$f(n_1 n_2 n_3 \ldots n_k) = \underline{1}^{n_1} 0\, 1^{n_2} 0 \ldots 0\, \underline{1}^{n_k} 0$$

$$f(n_1 n_2 \ldots n_k) = f(m_1 m_2 \ldots m_\ell)$$

$$\underline{1}^{n_1} 0\, \underline{1}^{n_2} 0 \ldots 0\, 1^{n_k} 0 = \underline{1}^{m_1} 0\, \underline{1}^{m_2} \ldots 0\, 1^{m_\ell} 0.$$

$$k = \ell.$$

and $n_1 = m_1$

$n_2 = m_2$

$\vdots$

$n_k = m_\ell$.

To prove B is countably infinite:

a) Write $B = \cup_{i>0} L_i$, where $|L_i|$ is finite.

b) Arrange elements of each L_i in a fixed order.

c) The first element of L_i has index $\sum_{j=1}^{i-1} |L_j|$.

e) We know position of $x \in L_i$.

$l=2$ $k=1$ (1)

$k=2$ 00

$l=3$ $k=1$ 20, 02, 11

$k=2$ 1

All $n_1 n_2 n_3 \ldots n_k \in S$ s.t. $n_1 + n_2 + \ldots + n_k + k = l$. such sequences are in L_l.

$$L_l = \{ n_1 n_2 \ldots n_k \mid \sum_{i=1}^{k} n_i + k = l \}$$

Each $|L_l|$ is finite.

$$S = \bigcup_{l \in \mathbb{N}} L_l$$

Question 3

$S \subseteq \mathbb{N}$

$S \in \mathcal{P}(\mathbb{N})$

$\mathcal{P}(\mathbb{N}) = \{ \emptyset, 0, 1 \ldots \ldots$

$\{0,1\}, \{0,2\} \ldots$

$\{0,1,2\} \ldots \}$

3. For an arbitrary set X, let $\mathcal{P}(X) = \{A : A \subseteq X\}$ denote the power set of X. Prove the following.
   (a) Show that $|\mathbb{N}| < |\mathcal{P}(\mathbb{N})|$.
   (b) Prove that there is no bijection $f : \mathcal{P}(X) \longrightarrow X$. Conclude that $|\mathcal{P}(X)| > |X|$.
   Hint: Prove by contradiction. Assume f is a bijection. Define $D_f \in \mathcal{P}(X)$ by $D_f = \{x \in X \mid x \notin f^{-1}(x)\}$.

$|\mathbb{N}| \leq |\mathcal{P}(\mathbb{N})|$ and $|\mathbb{N}| \neq |\mathcal{P}(\mathbb{N})|$

$|\mathcal{P}(X)| > |X|$

$f : \mathbb{N} \to \mathcal{P}(\mathbb{N})$ (one-one)

$f(n) = \{n\}$

$f(n) = f(m)$

$\Rightarrow \{n\} = \{m\} \Rightarrow n = m$

Suppose $|\mathcal{P}(\mathbb{N})| = |\mathbb{N}|$

$\Rightarrow g$ that is onto.

$D_N = \{ n \notin g(n) \}$